The Maximal Subword Complexity of Quasiperiodic Infinite Words

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We provide an exact estimate on the maximal subword complexity for quasiperiodic infinite words. To this end we give a representation of the set of finite and of infinite words having a certain quasiperiod q via a finite language derived from q. It is shown that this language is a suffix code having a bounded delay of decipherability.

Our estimate of the subword complexity now follows from this result, previously known results on the subword complexity and elementary results on formal power series.

Keywords: quasiperiodic words, codes, subword complexity, structure generating function

In his tutorial [Mar04] Solomon Marcus provided some initial facts on quasiperiodic infinite words. Here he posed several questions on the complexity of quasiperiodic infinite words. Some answers mainly for questions concerning quasiperiodic infinite words of low complexity were given in [LR04, LR07].

The investigations of the present paper turn to the question which are the maximally possible complexity functions for those words. As complexity we follow Marcus' [Mar04] Question 2 to consider the (subword) complexity function $f(\xi,n)$ of an infinite word ξ; $f(\xi,n)$ being its number of subwords of length n. This subword complexity of infinite words (ω-words) was mainly investigated for those words of low (polynomial) complexity (see the tutorial [BK03] or the book [AS03]). In [Sta93, Sta97] some results on exponential subword complexity helpful for the present considerations are derived.

As a final result we obtain that the maximally possible complexity functions for quasiperiodic infinite words ξ are bounded from above by a function of the form $f(\xi,n) \leq c_\xi \cdot t_P^n$ where t_P is the smallest Pisot-Vijayaraghavan number, that is, the unique real root t_P of the cubic polynomial $x^3 - x - 1$, which is approximately equal to $t_P \approx 1.324718$. We show also that this bound is tight, that is, there are ω-words ξ having $f(\xi,n) \approx c \cdot t_P^n$.

The paper is organised as follows. After introducing some notation we derive in Section 2 a characterisation of quasiperiodic words and ω-words having a certain quasiperiod q. Moreover, we introduce a finite basis set P_q from which the sets of quasiperiodic words or ω-words having quasiperiod q can be constructed. In Section 3 it is then proved that the star root of P_q is a suffix code having a bounded delay of decipherability.

This much prerequisites allow us, in Section 4 to estimate the number of subwords of the language Q_q of all quasiperiodic words having quasiperiod q. It turns out that $c_{q,1} \cdot \lambda_q^n \leq f(Q_q,n) \leq c_{q,2} \cdot \lambda_q^n$ where $f(Q_q,n)$ is the number of subwords of length n of words in Q_q and $1 \leq \lambda_q \leq t_P$ depends on q. From these results we derive our estimates for the subword complexity of quasiperiodic infinite words. Finally, we show that, for every quasiperiod q, there is a quasiperiodic ω-word ξ with quasiperiod q whose subword complexity $f(\xi,n)$ meets the upper bound $c_{q,2} \cdot \lambda_q^n$.

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1 Notation

In this section we introduce the notation used throughout the paper. By $\mathbb{N} = \{0,1,2,\ldots\}$ we denote the set of natural numbers. Let X be an alphabet of cardinality $|X| = r \geq 2$. By X^* we denote the set of finite words on X, including the *empty word* e, and X^ω is the set of infinite strings (ω-words) over X. Subsets of X^* will be referred to as *languages* and subsets of X^ω as ω-*languages*.

For $w \in X^*$ and $\eta \in X^* \cup X^\omega$ let $w \cdot \eta$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $L \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$. For a language L let $L^* := \bigcup_{i \in \mathbb{N}} L^i$, and by $L^\omega := \{w_1 \cdots w_i \cdots : w_i \in L \setminus \{e\}\}$ we denote the set of infinite strings formed by concatenating words in L. Furthermore $|w|$ is the *length* of the word $w \in X^*$ and $\mathbf{pref}(B)$ is the set of all finite prefixes of strings in $B \subseteq X^* \cup X^\omega$. We shall abbreviate $w \in \mathbf{pref}(\eta)$ $(\eta \in X^* \cup X^\omega)$ by $w \sqsubseteq \eta$.

We denote by $B/w := \{\eta : w \cdot \eta \in B\}$ the *left derivative* of the set $B \subseteq X^* \cup X^\omega$. As usual, a language $L \subseteq X^*$ is *regular* provided it is accepted by a finite automaton. An equivalent condition is that its set of left derivatives $\{L/w : w \in X^*\}$ is finite.

The sets of infixes of B or η are $\mathbf{infix}(B) := \bigcup_{w \in X^*} \mathbf{pref}(B/w)$ and $\mathbf{infix}(\eta) := \bigcup_{w \in X^*} \mathbf{pref}(\{\eta\}/w)$, respectively. In the sequel we assume the reader to be familiar with basic facts of language theory.

As usual a language $L \subseteq X^*$ is called a *code* provided $w_1 \cdots w_l = v_1 \cdots v_k$ for $w_1, \ldots, w_l, v_1, \ldots, v_k \in L$ implies $l = k$ and $w_i = v_i$.

2 Quasiperidicity

2.1 General properties

A finite or infinite word $\eta \in X^* \cup X^\omega$ is referred to as *quasiperiodic* with quasiperiod $q \in X^* \setminus \{e\}$ provided for every $j < |\eta| \in \mathbb{N} \cup \{\infty\}$ there is a prefix $u_j \sqsubseteq \eta$ of length $j - |q| < |u_j| \leq j$ such that $u_j \cdot q \sqsubseteq \eta$, that is, for every $w \sqsubseteq \eta$ the relation $u_{|w|} \sqsubset w \sqsubseteq u_{|w|} \cdot q$ is valid.

Let for $q \in X^* \setminus \{e\}$, Q_q be the set of quasiperiodic words with quasiperiod q. Then $\{q\}^* \subseteq Q_q = Q_q^*$ and $Q_q \setminus \{e\} \subseteq X^* \cdot q \cap q \cdot X^*$.

Definition 1 A family $\left(w_i\right)_{i=1}^{\ell}$, $\ell \in \mathbb{N} \cup \{\infty\}$, of words $w_i \in X^* \cdot q$ is referred to as a *q-chain* provided $w_1 = q$, $w_i \sqsubset w_{i+1}$ and $|w_{i+1}| - |w_i| \leq |q|$.

It holds the following.

Lemma 2

1. *$w \in Q_q \setminus \{e\}$ if and only if there is a q-chain $\left(w_i\right)_{i=1}^{\ell}$ such that $w_\ell = w$.*
2. *An ω-word $\xi \in X^\omega$ is quasiperiodic with quasiperiod q if and only if there is a q-chain $\left(w_i\right)_{i=1}^{\infty}$ such that $w_i \sqsubset \xi$.*

Proof: It suffices to show how a family $\left(u_j\right)_{j=0}^{|\eta|-1}$ can be converted to a q-chain $\left(w_i\right)_{i=1}^{\ell}$ and vice versa.

Consider $\eta \in X^* \cup X^\omega$ and let $\left(u_j\right)_{j=0}^{|\eta|-1}$ be a family such that $u_j \cdot q \sqsubseteq \eta$ and $j - |q| < |u_j| \leq j$ for $j < |\eta|$.

Define $w_1 := q$ and $w_{i+1} := u_{|w_i|} \cdot q$ as long as $|w_i| < |\eta|$. Then $w_i \sqsubseteq \eta$ and $|w_i| < |w_{i+1}| = |u_{|w_i|} \cdot q| \leq |w_i| + |q|$. Thus $\left(w_i\right)_{i=1}^{\ell}$ is a q-chain with $w_i \sqsubseteq \eta$.

Conversely, let $\left(w_i\right)_{i=1}^{\ell}$ be a q-chain such that $w_i \sqsubseteq \eta$ and set

$$u_j := \max_{\sqsubseteq}\left\{w' : \exists i(w' \cdot q = w_i \wedge |w'| \leq j)\right\}, \text{ for } j < |\eta|.$$

By definition, $u_j \cdot q \sqsubseteq \eta$ and $|u_j| \leq j$. Assume $|u_j| \leq j - |q|$ and $u_j \cdot q = w_i$. Then $|w_i| \leq j < |\eta|$. Consequently, in the q-chain there is a successor w_{i+1}, $|w_{i+1}| \leq |w_i| + |q| \leq j + |q|$. Let $w_{i+1} = w'' \cdot q$. Then $u_j \sqsubset w''$ and $|w''| \leq j$ which contradicts the maximality of u_j. □

Corollary 3 *Let $u \in \mathbf{pref}(Q_q)$. Then there are words $w, w' \in Q_q$ such that $w \sqsubseteq u \sqsubseteq w'$ and $|u| - |w|, |w'| - |u| \leq |q|$.*

Corollary 4 *Let $\xi \in X^\omega$. Then the following are equivalent.*

1. *ξ is quasiperiodic with quasiperiod q.*
2. *$\mathbf{pref}(\xi) \cap Q_q$ is infinite.*
3. *$\mathbf{pref}(\xi) \subseteq \mathbf{pref}(Q_q)$.*

2.2 A finite generator for quasiperiodic words

In this part we introduce the finite language P_q which generates the set of quasiperiodic words as well as the set of quasiperiodic ω-words having quasiperiod q. We investigate basic properties of P_q using simple facts from combinatorics on words (see e.g. [Shy01]). We set

$$P_q := \{v : e \sqsubset v \sqsubseteq q \sqsubset v \cdot q\}\,. \tag{1}$$

Then we have the following properties.

Proposition 5

$$\begin{aligned} Q_q &= P_q^* \cdot q \cup \{e\} \subseteq P_q^*\,, && (2)\\ \mathbf{pref}(P_q^*) &= \mathbf{pref}(Q_q) = P_q^* \cdot \mathbf{pref}(q) && (3) \end{aligned}$$

Proof: In order to prove Eq. (2) we show that $w_i \in P_q^* \cdot q$ for every q-chain $\left(w_i\right)_{i=1}^{\ell}$. This is certainly true for $w_1 = q$. Now proceed by induction on i. Let $w_i = w_i' \cdot q \in P_q^* \cdot q$ and $w_{i+1} = w_{i+1}' \cdot q$. Then $w_i' \cdot v_i = w_{i+1}'$. Now from $w_i \sqsubset w_{i+1}$ we obtain $e \sqsubset v_i \sqsubseteq q \sqsubset v_i \cdot q$, that is, $v_i \in P_q$.

Eq. (3) is an immediate consequence of Eq. (2). □

Corollary 4 and Proposition 5 imply the following characterisation of ω-words having quasiperiod q.

$$\{\xi : \xi \in X^\omega \wedge \xi \text{ has quasiperiod } q\} = P_q^\omega \tag{4}$$

Proof: Since P_q is finite, $P_q^\omega = \{\xi : \xi \in X^\omega \wedge \mathbf{pref}(\xi) \subseteq \mathbf{pref}(P_q^*)\}$. □

The following property of words in P_q is a consequence of the Lyndon-Schützenberger Theorem (see [BP85, Shy01]).

Proposition 6 *$v \in P_q$ if and only if $|v| \leq |q|$ and there is a prefix $\bar{v} \sqsubset v$ such that $q = v^k \cdot \bar{v}$ for $k = \lfloor |q|/|v| \rfloor$.*

Proof: Sufficiency is clear. Let now $v \in P_q$. Then $v \sqsubseteq q \sqsubset v \cdot q$. This implies $v^l \sqsubseteq q \sqsubset v^l \cdot q$ as long as $l \leq k$ and, finally, $q \sqsubset v^{k+1}$. □

Corollary 7 *$v \in P_q$ if and only if $|v| \leq |q|$ and there is a $k' \in \mathbb{N}$ such that $q \sqsubseteq v^{k'}$.*

Now set $q_0 := \min_{\sqsubseteq} P_q$. Then in view of Proposition 6 and Corollary 7 we have the following.

$$q = q_0^k \cdot \bar{q} \text{ for } k = \lfloor |q|/|q_0| \rfloor \text{ and some } \bar{q} \sqsubset q_0\,. \tag{5}$$

Corollary 8 *The word q_0 is primitive, that is, there are no $u \in X^*$ and $n > 1$ such that $q_0 = u^n$.*

Proof: Assume $q_0 = q_1^l$ for some $l > 1$. Then $\overline{q} = q_1^j \cdot \overline{q_1}$ where $\overline{q_1} \sqsubset q_1$, and, consequently, $q \sqsubset q_1^{k \cdot l + j + 1}$ contradicting the fact that q_0 is the shortest word in P_q. □

Proposition 9 *1. If $v \in P_q$ and $w \sqsubseteq q$ then $v \cdot w \sqsubseteq q$ or $q \sqsubseteq v \cdot w$.*

2. If $v \in P_q$ and $|v| \leq |q| - |q_0|$ then $v = q_0^m$ for some $m \in \mathbb{N}$.

Proof: The first assertion follows from $v \sqsubseteq q \sqsubset v \cdot q$ and $v \cdot w \sqsubseteq v \cdot q$.

For the proof of the second one observe that, by the first item $v \cdot q_0 \sqsubseteq q$ and $q_0 \cdot v \sqsubseteq q$ whence $q_0 \cdot v = v \cdot q_0$. Thus q_0 and v are powers of a common word. Since q_0 is primitive, the assertion follows. □

Theorem 10 *If $v \in P_q$ and $w \cdot v \sqsubseteq q$ then $w \in \{q_0\}^*$.*

Proof: If $v \in P_q$ then $q_0 \sqsubseteq v$. Thus it suffices to prove the assertion for q_0.

Let $w \cdot q_0 \sqsubseteq q = q_0^k \cdot \overline{q}$. Then $w \cdot q_0 \sqsubseteq q_0^{k+2}$ and, trivially, $q_0 \sqsubseteq q_0^{k+2}$. Since $|w \cdot q_0| + |q_0| < |q_0^{k+2}|$, $w \cdot q_0$ and q_0 are powers of a common word. The assertion follows because q_0 is primitive. □

3 Codes

In this section we investigate in more detail the properties of the star root of P_q, that is, of the smallest subset $V \subseteq P_q$ such that $V^* = P_q^{*q}$. It turns out that $\sqrt[*]{P_q}$ is a suffix code which, additionally, has a bounded delay of decipherability. This delay is closely related to the largest power of q_0 being a prefix of q.

According to [BP85] a subset $C \subseteq X^*$ is a code of a *delay of decipherability* $m \in \mathbb{N}$ if and only if for all $w, w', v_1, \ldots, v_m \in C$ and $u \in C^*$ the relation $w \cdot v_1 \cdots v_m \sqsubseteq w' \cdot u$ implies $w = w'$. Observe that $C \subseteq X^* \setminus \{e\}$ is a prefix code, that is, $w, w', \in C$ and $w \sqsubseteq w'$ imply $w = w'$, if and only if C has delay 0. A subset $C \subseteq X^* \setminus \{e\}$ is referred to as a *suffix code* if no word $w \in C$ is a proper suffix of another word $v \in C$.

Define now the *star-root* of P_q:

$$\sqrt[*]{P_q} := P_q \setminus \left(P_q^2 \cdot P_q^*\right)$$

It holds the following.

$$\sqrt[*]{P_q} = \left(P_q \setminus \{q_0\}^*\right) \cup \{q_0\} \subseteq \{q_0\} \cup \{v : v \sqsubseteq q \wedge |q_0| + |v| > |q|\} \tag{6}$$

Proof: First we prove the identity. The inclusion "$\subseteq$" follows from $\left(P_q \setminus \{q_0\}^*\right) \cup \{q_0\} \subseteq P_q \subseteq \left((P_q \setminus \{q_0\}^*) \cup \{q_0\}\right)^*$.

To prove the reverse inclusion assume $\ell > 1$ and $v_1 \cdots v_\ell \in P_q$ for $v_i \in P_q$. Then $|q_0| \leq |v_i|$ and thus $|q_0| + |v_i| \leq |q|$ for all i. According to Proposition 9.2 we have $v_i \in \{q_0\}^*$ which shows $P_q \cap \left(P_q^2 \cdot P_q^*\right) \subseteq \{q_0\}^*$.

The remaining inclusion now follows from Proposition 9.2. □

Next we are going to show that $\sqrt[*]{P_q}$ is a suffix code having a bounded delay of decipherability.

Corollary 11 *$\sqrt[*]{P_q}$ is a suffix code.*

Proof: Assume $u = w \cdot v$ for some $u, v \in \sqrt[*]{P_q}$, $u \neq v$. Then Theorem 10 proves $w \in \{q_0\}^* \subseteq P_q$. If $w \neq e$, in view of $u \sqsubseteq q$ Proposition 9.2 implies $v \in \{q_0\}^*$ and hence $u \in \{q_0\}^*$. Thus $u = v = q_0$ contradicting $u \neq v$. □

Theorem 12 *Let $q = q_0^k \cdot \bar{q}$ where $\bar{q} \sqsubset q_0$. Then $\sqrt[*]{P_q}$ is a code having a delay of decipherability of at most $k+1$.*

Proof: We have to show that if the words $v \cdot w_1 \cdots w_{k+1}$ and $v' \cdot w_1' \cdots w_{k+1}'$, where $v, w_1, \ldots, w_{k+1}, v', w_1', \ldots, w_{k+1}' \in \sqrt[*]{P_q}$ are comparable w.r.t. "$\sqsubseteq$" then $v = v'$.

Without loss of generality, assume $v \sqsubset v'$. Then $|q_0| \le |v| < |v'| \le |q|$. We have $|w_i|, |w_i'| \ge |q_0|$. Thus $|w_1 \cdots w_{k+1}|, |w_1' \cdots w_{k+1}'| > |q|$. Moreover, according to Proposition 9.1 $q \sqsubseteq w_1 \cdots w_{k+1}$ and $q \sqsubseteq w_1' \cdots w_{k+1}'$, whence $v \cdot q \sqsubset v' \cdot q$. Then in view of the inequality $|v| + |q| \ge |v'| + |q_0|$ we have $q \sqsupseteq w \cdot q_0$ for the word $w \ne e$ with $v \cdot w = v'$ and, according to Theorem 10 $w \in \{q_0\}^*$. This contradicts the fact that $\sqrt[*]{P_q}$ is a suffix code. □

We provide examples that, on the one hand, the bound in Theorem 12 cannot be improved and, on the other hand that it is not always attained. Since for $q = q_0^k$, $k \in \mathbb{N}$, the code $\sqrt[*]{P_q} = \{q_0\}$ is a prefix code, we consider only non-trivial cases.

Example 13 Let $q := aabaaaaba$. Then $q_0 = aabaa$, $k = 1$ and $\sqrt[*]{P_q} = P_q = \{\, q_0, aabaaaab, q \,\}$ which is a code having a delay of decipherability 2.

Indeed

$$\begin{array}{rcccl} aabaaaabaa & = & q_0 \cdot q_0 & \sqsubseteq & q \cdot q_0 \quad \text{or} \\ aabaaaabaa & = & q_0 \cdot q_0 & \sqsubseteq & aabaaaab \cdot q_0 \,. \end{array}$$

□

Moreover $q \cdot q_0 \notin Q_q$. Thus our Example 13 shows also that $q \cdot P_q^*$ need not be contained in Q_q.

Example 14 Let $q := aba$. Then $k = 1$ and $P_q = \{ab, aba\}$ is a code having a delay of decipherability 1. □

4 Subword Complexity

In this section we investigate the subword complexity of the language Q_q. To this end we derive general relations between the numbers of words of a certain length for regular languages, their prefix- and their infix-languages. Then using elementary methods of the theory of formal power series (cf. [BP85, SS78]) we estimate values characterising the exponential growth of the family $(|\mathbf{infix}(Q_q) \cap X^n|)_{n \in \mathbb{N}}$.

We start with some prerequisites on the number of subwords of regular star-languages.

Lemma 15 *If $L \subseteq X^*$ is a regular language then there is a $k \in \mathbb{N}$ such that*

$$\begin{array}{rclcl} |L \cap X^n| & \le & |\mathbf{pref}(L) \cap X^n| & \le & \sum_{i=0}^{k} |L \cap X^{n+i}| \\ |\mathbf{pref}(L) \cap X^n| & \le & |\mathbf{infix}(L) \cap X^n| & \le & k \cdot |\mathbf{pref}(L) \cap X^n| \end{array} \tag{7}$$

As a suitable k one may choose the number of states of an automaton accepting the language $L \subseteq X^*$.

Moreover, Corollary 4 of [Sta85] shows that for every regular language $L \subseteq X^*$ there are constants $c_1, c_2 > 0$ and a $\lambda \ge 1$ such that

$$c_1 \cdot \lambda^n \le |\mathbf{pref}(L^*) \cap X^n| \le c_2 \cdot \lambda^n \,. \tag{8}$$

A consequence of Lemma 15 is that Eq. (8) holds also (with constant $k \cdot c_2$ instead of c_2) for $\mathbf{infix}(L^*)$.

4.1 The subword complexity of Q_q

It is now our task to estimate the value λ_q which satisfies $c_1 \cdot \lambda_q^n \le |\mathbf{infix}(P_q^*) \cap X^n| \le k \cdot c_2 \cdot \lambda_q^n$. Following Lemma 15 and Eqs. (8) and (3) it holds

$$\lambda_q = \limsup_{n \to \infty} \sqrt[n]{|P_q^* \cap X^n|} \tag{9}$$

which is the inverse of the convergence radius $\mathtt{rad}\,\mathfrak{s}_q^*$ of the power series $\mathfrak{s}_q^*(t) := \sum_{n\in\mathbb{N}} |P_q^* \cap X^n| \cdot t^n$ (the structure generating function of the language P_q^*).

If $|q_0|$ divides $|q|$ then $P_q^* = \{q_0\}^*$ whence $\lambda_q = 1$. Therefore, in the following considerations we may assume that $|q|/|q_0| \notin \mathbb{N}$.

Since $\sqrt[*]{P_q}$ is a code, we have $\mathfrak{s}_q^*(t) = \frac{1}{1-\mathfrak{s}_q(t)}$ where $\mathfrak{s}_q(t) := \sum_{v\in \sqrt[*]{P_q}} t^{|v|}$ is the structure generating function of the finite language $\sqrt[*]{P_q}$. Thus the convergence radius $\mathtt{rad}\,\mathfrak{s}_q^*$ is the smallest root of $1-\mathfrak{s}_q(t)$. It is readily seen that this root is positive. So λ_q is the largest positive root of the reversed polynomial[1] $\mathfrak{p}_q(t) := t^{|q|} - \sum_{v\in \sqrt[*]{P_q}} t^{|q|-|v|}$. Summarising these observations we obtain the following.

Lemma 16 *Let $q \in X^* \setminus \{e\}$. Then there are constants $c_{q,1}, c_{q,2} > 0$ such that the structure function of the language $\mathbf{infix}(Q_q)$ satisfies*

$$c_{q,1} \cdot \lambda_q^n \leq |\mathbf{infix}(Q_q) \cap X^n| \leq c_{q,2} \cdot \lambda_q^n$$

where λ_q is the largest (positive) root of the polynomial $\mathfrak{p}_q(t)$.

Remark. One could prove Lemma 16 by showing that, for each polynomial $\mathfrak{p}_q(t)$, its largest (positive) root has multiplicity 1. Referring to Corollary 4 of [Sta85] (see Eq. (8)) we avoided these more detailed considerations of a particular class of polynomials.

In order to facilitate the search for the maximum of the values λ_q we may restrict our considerations to the case when $|q_0| > |q|/2$.

Lemma 17 *If $|q_0|$ does not divide $|q|$ and the language P_q^* is maximal w.r.t. "$\subseteq$" in the class $\left\{P_{q'}^* : q' \in X^* \setminus \{e\}\right\}$ then $|q_0| > |q|/2$.*

Proof: If $|q|/|q_0| \notin \mathbb{N}$ and $|q_0| \leq |q|/2$ we have $q = q_0^k \cdot \bar{q}$ for $k \geq 2$ and $e \neq \bar{q} \sqsubset q_0$. Then, obviously $P_q^* \subset P_{q'}^*$ for $q' := q_0 \cdot \bar{q}$. □

From $|q_0| > |q|/2$ we obtain that $\mathfrak{p}_q(t)$ has the form $t^{|q|} - \sum_{i\in M} t^i$ where $0 \in M \subseteq \{j : j < \frac{|q|}{2}\}$. In [Pol09] the following properties were derived.

Lemma 18 *Let $\mathscr{P} := \left\{t^n - \sum_{i\in M} t^i : n \geq 1 \wedge 0 \in M \subseteq \{j : j \leq \frac{n-1}{2}\}\right\}$. Then*

1. *for every $n \geq 1$ the polynomial $t^n - \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} t^i$ has the largest positive root among all polynomials of degree n in $\mathscr{P}$, and*
2. *the polynomials $t^3 - t - 1$ and $t^5 - t^2 - t - 1 = (t^2+1)\cdot(t^3-t-1)$ have the largest positive roots among all polynomials in $\mathscr{P}$.*

Two remarks are in order here.

1. It holds $\mathfrak{p}_{a^nba^n}(t) = t^{2n+1} - \sum_{i=0}^n t^i$ and $\mathfrak{p}_{a^nb^2a^n}(t) = t^{2n+2} - \sum_{i=0}^n t^i$, so for all degrees ≥ 1 there are polynomials of the form $\mathfrak{p}_q(t)$ in $\mathscr{P}$.
2. The positive root t_P of $\mathfrak{p}_{aba}(t) = t^3 - t - 1$ (or of $\mathfrak{p}_{a^2ba^2}(t)$) is known as the smallest Pisot-Vijayaraghavan number, that is, a positive root > 1 of a polynomial with integer coefficients all of whose conjugates have modulus smaller than 1.

Before proceeding to the proof of Lemma 18 we recall that the polynomials $p(t) \in \mathscr{P}$ have the following easily verified property.

$$\text{If } \varepsilon > 0 \text{ and } p(t') \geq 0 \text{ for some } t' > 0 \text{ then } p((1+\varepsilon)\cdot t') > 0\,. \tag{10}$$

[1] If $|q_0|$ divides $|q|$ we have $\mathfrak{p}_q(t) = t^{|q_0|} - 1$ instead.

Since $p(0) = -1 < 0$ for $p(t) \in \mathscr{P}$, Eq. (10) shows that once $p(t') \geq 0,\ t' > 0$ the polynomial $p(t)$ has no further root in the interval (t', ∞).

Proof: Using Eq. (10) the first assertion is easy to verify.

To show the second one it suffices to show that $p_n(t_P) > 0$ for every polynomial of the form $p_n(t) := t^n - \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} t^i$ other than $t^3 - t - 1$ or $t^5 - t^2 - t - 1$.

For degrees $n = 1, 2$ or $n = 4$ this is readily seen.

Now we proceed by induction on n. To this end we observe the following properties of the family $(p_n(t))_{n \geq 1}$.

$$p_{n+2}(t) - p_n(t) = t^{n+2} - t^n - t^{\lfloor \frac{n+1}{2} \rfloor} \text{ for } n \geq 3 \tag{11}$$

From this one easily obtains that $p_{n+2}(t_P) - p_n(t_P) = t_P^{n-1} - t_P^{\lfloor \frac{n+1}{2} \rfloor} > 0$ for $n \geq 4$, and the assertion follows by induction. □

4.2 The subword complexity of ω-words

Having derived the results on the the subword complexity of quasiperiodic words we are now in a position to contribute to an answer to Question 2 in [Mar04] by deriving tight upper bounds on the subword complexity of quasiperiodic infinite words.

To this end we recall that $\mathbf{infix}(\xi) \subseteq \mathbf{infix}(Q_q)$ for every ω-word ξ with quasiperiod q. Thus we obtain the following upper bound.

Lemma 19 *If $\xi \in X^\omega$ is quasiperiodic with quasiperiod q then $f(\xi, n) = |\mathbf{infix}(\xi) \cap X^n| \leq c \cdot \lambda_q^n$ for a suitable constant $c > 0$ not depending on ξ.*

Following the proof of Proposition 5.5 in [Sta93] it can be shown that this upper bound is tight.

Lemma 20 *For every quasiperiod $q \in X^* \setminus \{e\}$ there is a $\xi \in P_q^\omega$ such that $c_{q,1} \cdot \lambda_q^n \leq f(\xi, n) = |\mathbf{infix}(\xi) \cap X^n|$.*

Here $c_{q,1}$ is the constant mentioned in Lemma 16. *Proof:* Let $P_q^* = \{v_0, v_1, v_2 \ldots\}$ and define $\xi := \prod_{i \in \mathbb{N}} v_i$. Then obviously $\mathbf{infix}(\xi) = \mathbf{infix}(P_q^*) = \mathbf{infix}(Q_q)$. □

An over-all upper bound on the subword complexity of quasiperiodic ω-words now follows from Lemma 18.

Theorem 21 *There is a constant $c > 0$ such that for every quasiperiodic ω-word $\xi \in X^\omega$ there is an $n_\xi \in \mathbb{N}$ such that $f(\xi, n) = |\mathbf{infix}(\xi) \cap X^n| \leq c \cdot t_P^n$ for all $n \geq n_\xi$.*

We conclude this section with the following remark.
Remark. Theorem 21 is independent of the size of the alphabet X. And indeed, quasiperiodic ω-words of maximal subword complexity have quasiperiods of the form *aba* or *aabaa*, $a, b \in X,\ a \neq b$ (see the remark after Lemma 18), thus consist of only two different letters.

5 Concluding Remark

In the present paper we investigated the maximally achievable subword complexity for quasiperiodic infinite words. It should be mentioned that using results of [Sta93] the bounds obtained here can be extended to the Kolmogorov complexity of infinite words.

In [Sta93, Section 5] the asymptotic subword complexity of an ω-word $\xi \in X^\omega$ was introduced as $\tau(\xi) := \lim_{n\to\infty} \frac{\log_{|X|} |\mathbf{infix}(\xi)\cap X^n|}{n}$ and it was shown that τ is an upper bound to the asymptotic upper and lower Kolmogorov complexities of infinite words:

$$\underline{\kappa}(\xi) \leq \kappa(\xi) \leq \tau(\xi)\,.$$

Moreover, from the results of [Sta93, Section 4] it follows that for every quasiperiodic word q there is a $\xi \in P_q^\omega$ such that $\underline{\kappa}(\xi) = \tau(\xi) = \log_{|X|} \lambda_q$, that is, a quasiperiodic ω-word having quasiperiod q of maximally possible asymptotic (lower) Kolmogorov complexity.

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